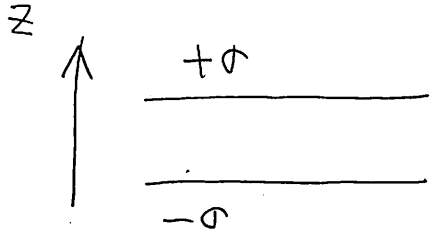
0.

(a) コンデンサーの2つの極板での表面電荷密度は $\pm\sigma$ $(\sigma = Q/S)$



左図のように z軸をとると*

$$\mathbf{D} = D\hat{z} \quad D = \sigma$$

電場は

$$\mathbf{E}_1 = E_1\hat{z} \quad E_1 = D/\epsilon_1 = \sigma/\epsilon_1$$
$$\mathbf{E}_2 = E_2\hat{z} \quad E_2 = D/\epsilon_2 = \sigma/\epsilon_2$$

(b) 電位差 V は

$$V = E_1 d_1 + E_2 d_2 = \sigma\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right)$$

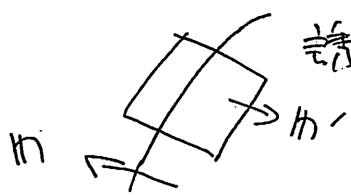
$$= \frac{Q}{S}\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right)$$

* 対称性からこの形を仮定し
(端の効果は無視)

$\nabla \cdot D = \sigma\delta(z)$ を極板付近で体積積分すればわかる。

1.

(a) 表面を含む微小領域δVで



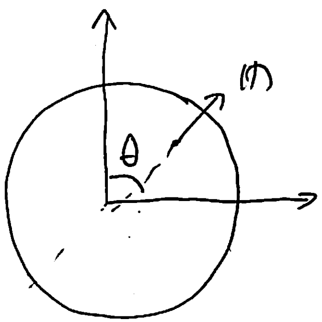
$$\int_{\delta V} dV\, \rho_p(\mathbf{r}) = -\int_{\delta V} dV\, \nabla\cdot\mathbf{P} = -\int_{\delta S} dS\, \mathbf{n}'\cdot\mathbf{P}$$

$$\mathbf{n}' = -\mathbf{n}$$

従って

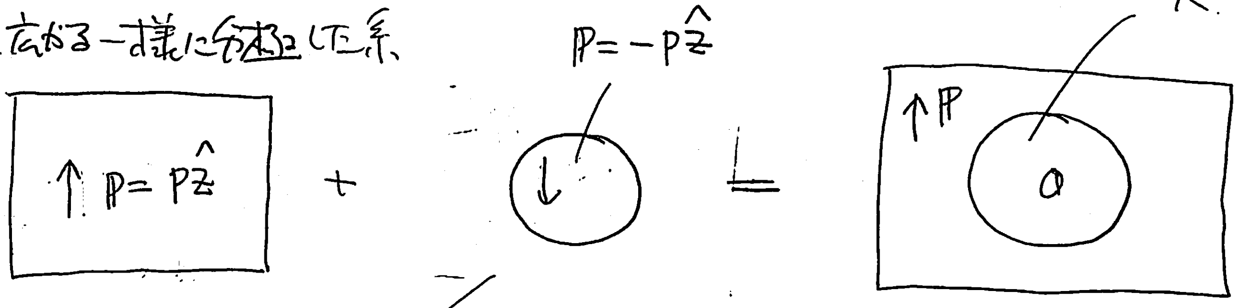
$$\sigma_p(\mathbf{r}) = \mathbf{n}(\mathbf{r})\cdot\mathbf{P}(\mathbf{r})$$

(b)

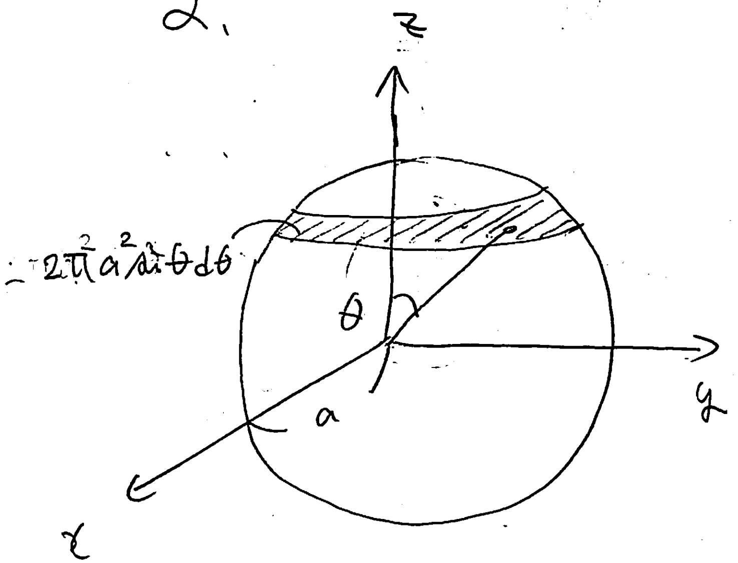


球座標系を用いると

$$\sigma_p = \mathbf{n}\cdot\mathbf{P} = P\cos\theta$$



2.



表面上の点 $(a\cos\theta,\ a\sin\theta\cos\varphi,\ a\sin\theta\sin\varphi)$

その表面分極電荷は $\sigma_p = \mathbb{n}\cdot(-p)\hat{z} = -p\cos\theta \qquad (\hat{\mathbb{n}} = \hat{r})$

これが穴の中につくる電場は

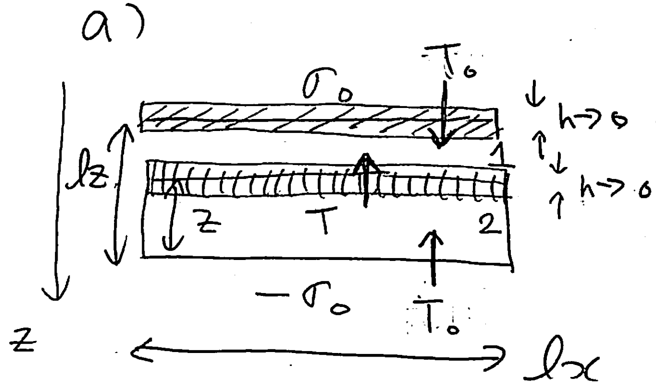
$$\mathbb{E}(0) = \frac{1}{4\pi\varepsilon_0}\int_0^{\pi} d\theta\ 2\pi a^2 \sin\theta\ \frac{-p\cos\theta}{a^2}\,(-\hat{r})$$

$\hat{r} = \cos\theta\,\hat{z} + \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y}$

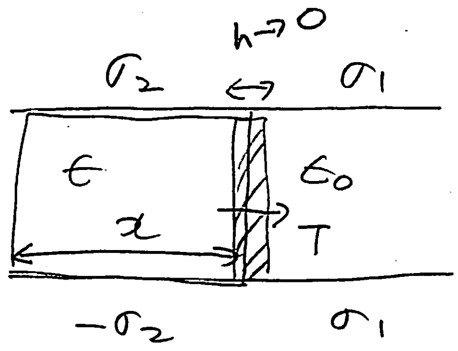
$$= \frac{p}{2\varepsilon_0}\int_0^{\pi} d\theta\ \cos^2\theta\sin\theta\ \hat{z} = \frac{p}{3\varepsilon_0}\hat{z}$$

3

a)



b)



(1) $\mathbf{E}=E_0\hat{z}$ $\quad E_0=\sigma_0/\epsilon_0$ $\quad$ 電位差は $\Delta\phi=E_0 l_z=\frac{\sigma_0}{\epsilon_0}l_z=\frac{Q}{\epsilon_0}\frac{l_z}{l_x l_y}$

よって 静電容量は $C=Q/\Delta\phi=\epsilon_0\frac{l_x l_y}{l_z}$

静電場のエネルギーは

$$U_0=\frac{1}{2}\epsilon_0 E_0^2 l_x l_y l_z=$$

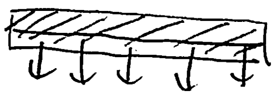
$$F_z=-\frac{\partial U_0}{\partial l_z}=-\frac{1}{2}\epsilon_0 E_0^2 l_x l_y$$

単位面積あたりに直すと $T_0=\frac{F_z}{l_x l_y}=-\frac{\epsilon_0}{2}E_0^2$

向きは図の向き。

注 Maxwell応力は考える領域の外向き法線ベクトルをもとに定義されている。

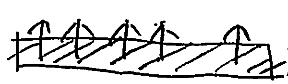
$E=0$



極板

$$T_{zz}=\epsilon_0\left(E_0^2-\frac{E_0^2}{2}\right)=\frac{\epsilon_0}{2}E_0^2$$

$E=E_0$



極板

Maxwell応力の表式で図の [hatched] のように極板を含む無限小の厚み $(h\to 0)$ の領域に用いれば

$$T_0=-\frac{1}{2}\epsilon_0 E^2$$ で一致する

(2) $D_1 = D_2$ (境界面に垂直なので) 従って $\mathbf{E}_1 = E_1\hat{z}$ $\quad \mathbf{E}_2 = E_2\hat{z}$ $\quad E_1 = \sigma/\epsilon_0 =$

$E_2 = \sigma/\epsilon =$

電位差は $\Delta\phi = E_1(l_z - z) + E_2 z$

$$= \sigma\left[\frac{l_z}{\epsilon_0} + \left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right)z\right] = \sigma\frac{l_z}{\epsilon_0}\left[1 - \left(1 - \frac{\epsilon_0}{\epsilon}\right)\frac{z}{l_z}\right]$$

よって 静電容量は、 $C = Q/\Delta\phi = C_0 \dfrac{1}{1 - \left(1 - \frac{\epsilon_0}{\epsilon}\right)\frac{z}{l_z}}$

静電場のエネルギーは、

$$U = \frac{1}{2}\epsilon_0 E_1^2 l_x l_y (l_z - z) + \frac{1}{2}\epsilon E_2^2 l_x l_y z$$

$$= \underbrace{\frac{1}{2}\epsilon_0 E_0^2 l_x l_y l_z}_{U_0}\left[1 - \frac{z}{l_z} + \underbrace{\frac{\epsilon}{\epsilon_0}\left(\frac{E_2}{E_1}\right)^2}_{\epsilon_0/\epsilon}\frac{z}{l_z}\right]$$

$$= U_0\left[1 - \left(1 - \frac{\epsilon_0}{\epsilon}\right)\frac{z}{l_z}\right]$$ (U_0よりも減少している)

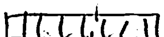
これから 境界面に働く力は

$$F_z = -\frac{\partial U}{\partial z} = U_0\left(1 - \frac{\epsilon_0}{\epsilon}\right)\frac{1}{l_z}$$

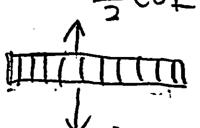
単位面積あたりで

$$T = \frac{F_z}{l_x l_y} = \frac{U_0}{V}\left(1 - \frac{\epsilon_0}{\epsilon}\right.$$

向きは 図の向き。

Maxwell応力の表式を図の  の領域に用いると

$\frac{1}{2}\epsilon_0 E_1^2$



$\frac{1}{2}\epsilon E_2^2$

$$T_{zz} = \frac{1}{2}\epsilon_0 E_1^2 - \frac{1}{2}\epsilon E_2^2 = \frac{1}{2}\epsilon_0 E_1^2\left(1 - \frac{\epsilon_0}{\epsilon}\right) = \frac{U_0}{V}\left(1 - \frac{\epsilon_0}{\epsilon}\right)$$ で一致

(3) 導体上は等電位になっている → 領域1,2で電場は同じ $\mathbb{E}_1 = \mathbb{E}_2 = E\hat{z}$

$\mathbb{E}_1 = \dfrac{\sigma_1}{\epsilon_0}\hat{z}$ $\mathbb{E}_2 = \dfrac{\sigma_2}{\epsilon}\hat{z}$ なので $\dfrac{\sigma_2}{\sigma_1} = \dfrac{\epsilon}{\epsilon_0}$ …①

また $\sigma_1\left(1-\dfrac{x}{l_x}\right) + \sigma_2\dfrac{x}{l_x} = \sigma_0$ (電荷の保存) …②

これから

$$\sigma_1 = \frac{\sigma_0}{1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}}$$

電場は $\mathbb{E}_1 = \mathbb{E}_2 = E\hat{z}$

$$E = E_0\frac{\sigma_1}{\sigma_0}$$

$$= E_0\frac{1}{1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}}$$

電位差は

$$\Delta\phi = E\,l_z = \frac{\sigma_1}{\epsilon_0}l_z$$

$$= \frac{\sigma_0}{\epsilon_0}l_z\frac{1}{1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}}$$

静電容量は,

$$C = Q/\Delta\phi = C_0\left[1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}\right]$$

静電エネルギーは

$$U = \frac{1}{2}\epsilon E^2 x\,l_x l_y l_z + \frac{1}{2}\epsilon_0 E^2 (l_x - x) l_y l_z$$

$$= \underbrace{\frac{1}{2}\epsilon_0 E_0^2 l_x l_y l_z}_{U_0}\left(\frac{E}{E_0}\right)^2\left[\frac{\epsilon}{\epsilon_0}\frac{x}{l_x} + \left(1-\frac{x}{l_x}\right)\right]$$

$$E/E_0 = \sigma_1/\sigma_0 = \left[1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}\right]^{-1}$$

$$= \frac{U_0}{1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}}$$

境界面に働く力は

$$F_x = -\frac{\partial U}{\partial x} = \frac{U_0}{\left[1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}\right]^2}\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{1}{l_x}$$

単位面積あたりの大きさは

$$T = \frac{|F_x|}{l_y l_z} = \frac{U_0}{V_0}\,\frac{\frac{\epsilon}{\epsilon_0}-1}{\left[1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}\right]^2}$$

向きは 図の向…

($\epsilon > \epsilon_0$ならば

xを増やす方…

一方、Maxwell応力の表式を図の▨▨▨の領域($h\to 0$)に用いると

$$T_{xx} = \frac{1}{2}\epsilon E^2 - \frac{1}{2}\epsilon_0 E^2$$

$$= \frac{1}{2}\epsilon_0 E^2\left(\frac{\epsilon}{\epsilon_0}-1\right) = \frac{1}{2}\epsilon_0 E_0^2\,\frac{\frac{\epsilon}{\epsilon_0}-1}{\left[1+\left(\frac{\epsilon}{\epsilon_0}-1\right)\frac{x}{l_x}\right]^2}$$

で一致する。