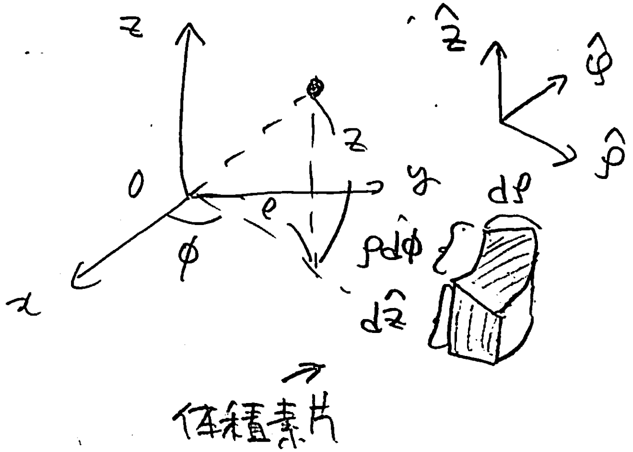
電磁気学演習 No7. 略解

0.

円筒座標系 (ρ, ϕ, z)

(a)



$h_\rho = 1 \quad h_\phi = \rho \quad h_z = 1$

(b)

$$\nabla \cdot A = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z}(\rho A_z)\right]$$

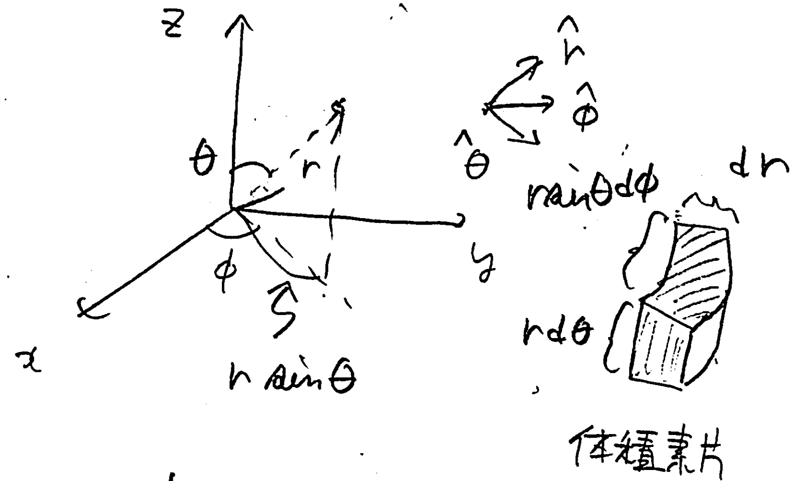
$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\Delta \psi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\nabla \times A = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \hat{\phi}\rho & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\hat{z}$$

球座標系 (r, θ, ϕ)



$h_r = 1 \quad h_\theta = r \quad h_\phi = r\sin\theta$

$$\nabla \cdot A = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(r A_\phi)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2 A_r) + r\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + r\frac{\partial A_\phi}{\partial \phi}\right]$$

$$\Delta \psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$\nabla \times A = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

1.

(a)

$$\nabla\cdot\mathbf{E} = 0 \quad \cdots ①$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \cdots ②$$

$$\nabla\cdot\mathbf{B} = 0 \quad \cdots ③$$

$$\nabla\times\mathbf{B} = \mu_0\left(\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \sigma\mathbf{E}\right) \quad \cdots ④$$

(b)

② の両辺の $\nabla\times\cdots$ をとって

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}$$

$$\nabla(\underbrace{\nabla\cdot\mathbf{E}}_{=0}) - \nabla^2\mathbf{E}$$

これに ④ を用いれば

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}$$

同様に ④ の両辺の $\nabla\times\cdots$ をとって

$$\nabla\times(\nabla\times\mathbf{B}) = \left(\mu_0\epsilon_0\frac{\partial}{\partial t} + \sigma\right)\nabla\times\mathbf{E}$$

$$\nabla(\underbrace{\nabla\cdot\mathbf{B}}_{=0}) - \nabla^2\mathbf{B}$$

これに ② を用いて

$$\nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t}$$

まとめると、$\mathbf{E}$, $\mathbf{B}$ いずれの成分も次の形の方程式に従う。

$$\nabla^2\psi - \mu_0\sigma\frac{\partial\psi}{\partial t} - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0$$

(c) $e^{i(kx-\omega t)}$ の形を仮定すると方程式に代入して

$$(ik)^2 + i\mu_0\sigma\omega - \mu\epsilon(i\omega)^2 = 0$$

すなわち $$k^2 = \mu\epsilon\omega^2 + i\mu_0\sigma\omega$$

ここで $k = \alpha + i\beta$ とおくと

$$\begin{cases} \alpha^2 - \beta^2 = \mu\epsilon\omega^2 \\ 2\alpha\beta = \mu_0\sigma\omega \end{cases}$$

これをとくと

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\mu\epsilon}\right)^2}+1\right]^{1/2}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\mu\epsilon}\right)^2}-1\right]^{1/2}$$

したがって

$$e^{i(kx-\omega t)} = e^{i(\alpha x-\omega t)} \cdot e^{-\beta x}$$

で振幅は指数関数的に減衰する.

2.

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \quad \cdots (\#)$$

(a)

$$\nabla\psi = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r},\ \frac{\partial r}{\partial y}\frac{\partial}{\partial r},\ \frac{\partial r}{\partial z}\frac{\partial}{\partial r}\right)\psi = \left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right)\frac{\partial\psi}{\partial r}$$

$$= \hat{r}\frac{\partial\psi}{\partial r}$$

$$\nabla\cdot\nabla\psi = \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\cdot\left\{(x, y, z)\ \frac{1}{r}\frac{\partial\psi}{\partial r}\right\}$$

$$= 3\frac{1}{r}\frac{\partial\psi}{\partial r} + \left(x\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + y\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + z\frac{\partial r}{\partial z}\frac{\partial}{\partial r}\right)\frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$= 3\frac{1}{r}\frac{\partial\psi}{\partial r} + \left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right)\left(-\frac{1}{r^2}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right)$$

$$= 2\frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

(b) 元の方程式 (#) は

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

つまり

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

これは 1次元の 波動方程式である。従って一般解として

$$\psi(r,t) = \frac{1}{r}\left(f(r-ct) + g(r+ct)\right)$$ の形が得られる。