線形代数 I (担当 松下勝義)

VII. 余因子展開と逆行列, 行列式の演算

教科書 §4.3-4.6, pp.53–75

- 余因子展開

多重線形性を考えるとある行を次のように分解できる.

余因子展開

$$
\begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{i-11} & a_{i-12} & \dots & a_{i-1n} \\
a_{i1} & a_{i2} & \dots & a_{in} \\
a_{i+11} & a_{i+12} & \dots & a_{i+1n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{vmatrix}
= \sum_j a_{ij} (-1)^(i+j)
\begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1j-1} & a_{1j+1} & \dots & a_{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{i-11} & a_{i-12} & \dots & a_{i-1j-1} & a_{1-1j+1} & \dots & a_{i-1n} \\
a_{i+11} & a_{i+12} & \dots & a_{i+1n} & a_{i+1j+1} & \dots & a_{i+1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \dots & a_{nj-1} & a_{nj+1} & \dots & a_{nn}
\end{vmatrix}
\tag{463}
$$

余因子展開の例を挙げると,

– n=2 の場合，
多重線形性から

$$
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \tag{464}
$$

$$
= \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + (-1) \begin{vmatrix} a_{12} & 0 \\ a_{22} & a_{21} \end{vmatrix} \tag{465}
$$

$$
= a_{11} \begin{vmatrix} a_{22} \end{vmatrix} + (-1) a_{12} \begin{vmatrix} a_{21} \end{vmatrix} \tag{466}
$$

$$
= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} \end{vmatrix} \tag{467}
$$

$$
= \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{vmatrix} \tag{468}
$$

$$
= (-1) \begin{vmatrix} 0 & a_{22} \\ a_{11} & a_{12} \end{vmatrix} + (-1) \begin{vmatrix} a_{21} & 0 \\ a_{12} & a_{11} \end{vmatrix} \tag{469}
$$

$$
= (-1)^2 \begin{vmatrix} a_{22} & 0 \\ a_{12} & a_{11} \end{vmatrix} + (-1) \begin{vmatrix} a_{21} & 0 \\ a_{12} & a_{11} \end{vmatrix} \tag{470}
$$

$$
= a_{22} (-1)^2 \begin{vmatrix} a_{11} \end{vmatrix} + (-1) a_{21} \begin{vmatrix} a_{12} \end{vmatrix} \tag{471}
$$

$$
= (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} \end{vmatrix} + (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} \end{vmatrix} \tag{472}
$$

– n=3 の場合,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a13 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{473}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + (-1) \begin{vmatrix} a_{12} & 0 & 0 \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} + (-1) \begin{vmatrix} 0 & a_{13} & 0 \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{vmatrix} \tag{474}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + (-1) \begin{vmatrix} a_{12} & 0 & 0 \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} + (-1)^2 \begin{vmatrix} a_{13} & 0 & 0 \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{vmatrix} \tag{475}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^2 a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{476}$$

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{477}$$

などと展開できる.

余因子

余因子展開を

$$\left| \hat{A} \right| = \sum_j a_{ij} \tilde{a}_{ij} \tag{478}$$

と書いたとき, n-1 行 n-1 列行列の行列式 $\tilde{a}_{ij}$

$$\tilde{a}_{ij} = (-1)^(i+j) \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j-1} & a_{1j+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-11} & a_{i-12} & \dots & a_{i-1j-1} & a_{1-1j+1} & \dots & a_{i-1n} \\ a_{i+11} & a_{i+12} & \dots & a_{i+1n} & a_{i+1j+1} & \dots & a_{i+1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & a_{nj+1} & \dots & a_{nn} \end{vmatrix} \tag{479}$$

を余因子と呼ぶ.

余因子に対して以下の性質がある.

余因子の性質

行列 $\hat{A}$ の i 行目の余因子を列に並べたベクトル

$$\tilde{\boldsymbol{a}}_i = \begin{pmatrix} \tilde{a}_{i1} \\ \tilde{a}_{i2} \\ \vdots \\ \tilde{a}_{in} \end{pmatrix} \tag{480}$$

に対して j 行ベクトル $\boldsymbol{a}_j$ を考えると

$$\begin{cases} i = j & \boldsymbol{a}_j \tilde{\boldsymbol{a}}_i = \boldsymbol{a}_i \tilde{\boldsymbol{a}}_i = \sum_k a_{ik}\tilde{a}_{ik} = \left|\hat{A}\right| \\ i \neq j & \boldsymbol{a}_j \tilde{\boldsymbol{a}}_i = \sum_k a_{jk}\tilde{a}_{ik} = \begin{vmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{i-1} \\ \boldsymbol{a}_j \\ \boldsymbol{a}_{i+1} \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} = 0 \end{cases} \tag{481}$$

従って, 余因子行列 $\hat{\tilde{A}}$

$$\hat{\tilde{A}} = \begin{pmatrix} \tilde{\boldsymbol{a}}_1 & \tilde{\boldsymbol{a}}_2 & \ldots & \tilde{\boldsymbol{a}}_n \end{pmatrix} \tag{482}$$

を考えると,

$$\hat{A}\hat{\tilde{A}} = \begin{pmatrix} \left|\hat{A}\right| & 0 & \ldots & 0 \\ 0 & \left|\hat{A}\right| & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \left|\hat{A}\right| \end{pmatrix} = \left|\hat{A}\right| \hat{I}_n \tag{483}$$

となる. そのため次の逆行列の公式 (Cramer's rule) が成り立つ.

逆行列の公式

$$\hat{A}^{-1} = \frac{\hat{\tilde{A}}}{\left|\hat{A}\right|} \tag{484}$$

- 行列式の積

行列式の積

次元が同じ正方行列 $\hat{A}$ と $\hat{B}$ の行列式の積にたいして,

$$\left|\hat{A}\hat{B}\right| = \left|\hat{A}\right|\left|\hat{B}\right| \tag{485}$$

のように $|\hat{A}|$ とに $|\hat{B}|$ の積になる.

- 逆行列の行列式

逆行列の行列式

正則行列 $\hat{A}$ の逆行列 $\hat{A}^{-1}$ の行列式は,

$$\left|\hat{A}^{-1}\right| = \left|\hat{A}^{-1}\right| \frac{\left|\hat{A}\right|}{\left|\hat{A}\right|} = \frac{\left|\hat{A}^{-1}\hat{A}\right|}{\left|\hat{A}\right|} = \frac{1}{\left|\hat{A}\right|} \tag{486}$$

なので $|\hat{A}|$ の逆行列になる.