

## 補足8 余因子展開と逆行列

### 1. 行列式の多重線形性

$k$  と  $l$  をある定数としたとき次の行列式の関係式が成立する.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} + lb_{i1} & ka_{i2} + lb_{i2} & \cdots & ka_{in} + lb_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + l \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (201)$$

#### ● 2 次の正方行列の例

$$\begin{vmatrix} 1 & 2 \\ 2 \times 1 + 3 \times 1 & 2 \times 2 + 3 \times 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \quad (202)$$

右辺と左辺はともに 3 である.

#### ● 2 次の正方行列の例 2

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \\ = (a_{11}a_{22} - 0 \times a_{21}) + (0 \times a_{22} - a_{12}a_{21}) \\ = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (203)$$

### 2. 1 行目の余因子展開

$n$  次正方行列  $\hat{A} = (a_{ij})$  の行列式は定義より,

$$|\hat{A}| = \sum_{\sigma \in P_n} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}. \quad (204)$$

これは 1 行目の成分を和の前に出し,  $\text{sgn}(\sigma)$  でも一番先頭へ持ってくることで次のようにも書ける.,

$$\begin{aligned} |\hat{A}| &= \sum_{j=1}^n a_{1j} (-1)^{j-1} \sum_{\sigma' \in P_{(n-1)}} \text{sgn}(\sigma') a_{2\sigma'(2)} a_{3\sigma'(3)} \cdots a_{n\sigma'(n)} \\ &= \sum_{j=1}^n a_{1j} \tilde{a}_{1j} \end{aligned} \quad (205)$$

この展開式を余因子展開と呼ぶ. ここで

$$\begin{aligned} \tilde{a}_{1j} &= (-1)^{j-1} \sum_{\sigma' \in P_{(n-1)}} \text{sgn}(\sigma') a_{2\sigma'(2)} a_{3\sigma'(3)} \cdots a_{n\sigma'(n)} \\ &= (-1)^{j-1} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j-1} & a_{3j+1} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix} \end{aligned} \quad (206)$$

を余因子と呼ぶ.

- 例 2 次の正方行列

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}(-1)^{1-1}|a_{22}| + a_{12}(-1)^{2-1}|a_{21}| \\ &= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} \end{aligned} \quad (207)$$

ただし,

$$\tilde{a}_{11} = (-1)^{1-1}|a_{22}| \quad (208)$$

$$\tilde{a}_{12} = (-1)^{2-1}|a_{21}| \quad (209)$$

- 例 3 次の正方行列

行列の多重線形性を用いて,

$$\begin{aligned}
 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
 &= (-1)^{1-1} \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + (-1)^{2-1} \begin{vmatrix} a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
 &\quad + (-1)^{3-1} \begin{vmatrix} a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
 &= (-1)^{1-1} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} + (-1)^{2-1} \begin{vmatrix} a_{12} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} \\
 &\quad + (-1)^{3-1} \begin{vmatrix} a_{13} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} \\
 &= a_{11}(-1)^{1-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} + a_{12}(-1)^{2-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \\
 &\quad + a_{13}(-1)^{3-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \\
 &= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} \tag{210}
 \end{aligned}$$

### 3. $i$ 行目の余因子展開

$i$  行目に対しても同じように余因子展開ができる. これは  $i$  行目の成分を和の前に出し,  $\text{sgn}(\sigma)$  でも一番先頭へ持ってくることで次のよう書くことで行える.

$$|\hat{A}| = \sum_{j=1}^n a_{ij} \tilde{a}_{ij} \quad (211)$$

ここで

$$\begin{aligned} \tilde{a}_{ij} &= (-1)^{i+j} \sum_{\sigma' \in P_{(n-1)}} \text{sgn}(\sigma') a_{2\sigma'(2)} a_{3\sigma'(3)} \cdots a_{n\sigma'(n)} \\ &= (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1j-1} & a_{i-1j+1} & \cdots & a_{i-1n} \\ a_{i+11} & a_{i+12} & \cdots & a_{i+1j-1} & a_{i+1j+1} & \cdots & a_{i+1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix} \end{aligned}$$

#### • 例 2 次の正方行列

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{22}(-1)^{2+2}|a_{11}| + a_{21}(-1)^{2+1}|a_{12}| \\ &= a_{22}\tilde{a}_{22} + a_{21}\tilde{a}_{21} \end{aligned} \quad (212)$$

$$= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} \quad (213)$$

ただし,

$$\tilde{a}_{21} = (-1)^{2+1}|a_{12}| \quad (214)$$

$$\tilde{a}_{22} = (-1)^{2+2}|a_{22}| \quad (215)$$

#### • 例 3 次の正方行列

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} \\ &= a_{21}\tilde{a}_{21} + a_{22}\tilde{a}_{22} + a_{23}\tilde{a}_{23} \\ &= a_{31}\tilde{a}_{31} + a_{32}\tilde{a}_{32} + a_{33}\tilde{a}_{33} \end{aligned}$$

#### 4. 余因子と逆行列

$n$  次正方行列  $\hat{A}$  が正則ならば逆行列  $A^{-1}$  は余因子を用いて次のように書ける.

$$\hat{A}^{-1} = \frac{\hat{A}}{|\hat{A}|} \quad (216)$$

ただし  $\hat{A}$  は余因子行列であり,

$$\hat{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \cdots & \tilde{a}_{n1} \\ \tilde{a}_{12} & \tilde{a}_{22} & \cdots & \tilde{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1n} & \tilde{a}_{2n} & \cdots & \tilde{a}_{nn} \end{pmatrix} \quad (217)$$

である.

- 2 次の正則行列

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (218)$$

に対して, 次の行列を考える.

$$\begin{aligned} |\hat{A}|I_2 &= \begin{pmatrix} |\hat{A}| & 0 \\ 0 & |\hat{A}| \end{pmatrix} \\ &= \begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} & a_{11}\tilde{a}_{21} + a_{12}\tilde{a}_{22} \\ a_{21}\tilde{a}_{11} + a_{22}\tilde{a}_{12} & a_{21}\tilde{a}_{21} + a_{22}\tilde{a}_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} \\ \tilde{a}_{12} & \tilde{a}_{22} \end{pmatrix} \\ &= \hat{A}\hat{A} \end{aligned} \quad (219)$$

従って両辺を  $|\hat{A}|$  で割り左から  $\hat{A}^{-1}$  を掛けると

$$\hat{A} = \frac{\hat{A}}{|\hat{A}|} \quad (220)$$