補足 8　余因子展開と逆行列

1. 行列式の多重線形性
   k と l をある定数としたとき次の行列式の関係式が成立する.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ ka_{i1}+lb_{i1} & ka_{i2}+lb_{i2} & \cdots & ka_{in}+lb_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + l \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{201}$$

   - 2 次の正方行列の例

$$\begin{vmatrix} 1 & 2 \\ 2\times 1+3\times 1 & 2\times 2+3\times 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \tag{202}$$

   右辺と左辺はともに 3 である.

   - 2 次の正方行列の例 2

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= (a_{11}a_{22} - 0\times a_{21}) + (0\times a_{22} - a_{12}a_{21}) \\ &= \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{aligned} \tag{203}$$

2. 1 行目の余因子展開
   n 次正方行列 $\hat{A} = (a_{ij})$ の行列式は定義より,

$$|\hat{A}| = \sum_{\sigma\in P_n} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}. \tag{204}$$

   これは 1 行目の成分を和の前に出し, sgn(σ) でも一番先頭へ持ってくることで次のようにも書ける.,

$$\begin{aligned} |\hat{A}| &= \sum_{j=1}^{n} a_{1j}(-1)^{j-1} \sum_{\sigma'\in P_(n-1)} \mathrm{sgn}(\sigma') a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} \\ &= \sum_{j=1}^{n} a_{1j}\tilde{a}_{1j} \end{aligned} \tag{205}$$

この展開式を余因子展開と呼ぶ. ここで

$$
\begin{aligned}
\tilde{a}_{1j} &= (-1)^{j-1} \sum_{\sigma' \in P_(n-1)} \mathrm{sgn}(\sigma') a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} \\
&= (-1)^{j-1} \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2j-1} & a_{2j+1} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3j-1} & a_{3j+1} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & a_{nj+1} & \dots & a_{nn} \end{vmatrix} \qquad (206)
\end{aligned}
$$

を余因子と呼ぶ.

- 例 2 次の正方行列

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\
&= a_{11}(-1)^{1-1}|a_{22}| + a_{12}(-1)^{2-1}|a_{21}| \\
&= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} \qquad (207)
\end{aligned}
$$

ただし,

$$\tilde{a}_{11} = (-1)^{1-1}|a_{22}| \qquad (208)$$

$$\tilde{a}_{12} = (-1)^{2-1}|a_{21}| \qquad (209)$$

- 例 3 次の正方行列

行列の多重線形性を用いて,

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= (-1)^{1-1} \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + (-1)^{2-1} \begin{vmatrix} a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&+ (-1)^{3-1} \begin{vmatrix} a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= (-1)^{1-1} \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} + (-1)^{2-1} \begin{vmatrix} a_{12} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} \\
&+ (-1)^{3-1} \begin{vmatrix} a_{13} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} \\
&= a_{11}(-1)^{1-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} + a_{12}(-1)^{2-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \\
&+ a_{13}(-1)^{3-1} \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \\
&= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} \qquad (210)
\end{aligned}
$$

3. i 行目の余因子展開

i 行目に対しても同じように余因子展開ができる．これは i 行目の成分を和の前に出し, $\mathrm{sgn}(\sigma)$ でも一番先頭へ持ってくることで次のよう書くことで行える.,

$$|\hat{A}| = \sum_{j=1}^{n} a_{ij}\tilde{a}_{ij} \tag{211}$$

ここで

$$\tilde{a}_{ij} = (-1)^{i+j} \sum_{\sigma' \in P_(n-1)} \mathrm{sgn}(\sigma') a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}$$
$$= (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1j-1} & a_{1j+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j-1} & a_{2j+1} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-11} & a_{i-12} & \ldots & a_{i-1j-1} & a_{i-1j+1} & \ldots & a_{i-1n} \\ a_{i+11} & a_{i+12} & \ldots & a_{i+1j-1} & a_{i+1j+1} & \ldots & a_{i+1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nj-1} & a_{nj+1} & \ldots & a_{nn} \end{vmatrix}$$

- 例 2 次の正方行列

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
$$= a_{22}(-1)^{2+2}|a_{11}| + a_{21}(-1)^{2+1}|a_{12}|$$
$$= a_{22}\tilde{a}_{22} + a_{21}\tilde{a}_{21} \tag{212}$$
$$= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12}$$
$$\tag{213}$$

ただし,

$$\tilde{a}_{21} = (-1)^{2+1}|a_{12}| \tag{214}$$
$$\tilde{a}_{22} = (-1)^{2+2}|a_{22}| \tag{215}$$

- 例 3 次の正方行列

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13}$$
$$= a_{21}\tilde{a}_{21} + a_{22}\tilde{a}_{22} + a_{23}\tilde{a}_{23}$$
$$= a_{31}\tilde{a}_{31} + a_{32}\tilde{a}_{32} + a_{33}\tilde{a}_{33}$$

4. 余因子と逆行列

n 次正方行列 $\hat{A}$ が正則ならば逆行列 A^{-1} は余因子を用いて次のように書ける.

$$\hat{A}^{-1} = \frac{\hat{\tilde{A}}}{|\hat{A}|} \tag{216}$$

ただし $\hat{\tilde{A}}$ は余因子行列であり,

$$\hat{\tilde{A}} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \dots & \tilde{a}_{n1} \\ \tilde{a}_{12} & \tilde{a}_{22} & \dots & \tilde{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{1n} & \tilde{a}_{2n} & \dots & \tilde{a}_{nn} \end{pmatrix} \tag{217}$$

である.

- 2 次の正則行列

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{218}$$

に対して, 次の行列を考える.

$$\begin{aligned} |\hat{A}|I_2 &= \begin{pmatrix} |\hat{A}| & 0 \\ 0 & |\hat{A}| \end{pmatrix} \\ &= \begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} & a_{11}\tilde{a}_{21} + a_{12}\tilde{a}_{22} \\ a_{21}\tilde{a}_{11} + a_{22}\tilde{a}_{12} & a_{21}\tilde{a}_{21} + a_{22}\tilde{a}_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} \\ \tilde{a}_{12} & \tilde{a}_{22} \end{pmatrix} \\ &= \hat{A}\hat{\tilde{A}} \end{aligned} \tag{219}$$

従って両辺を $|\hat{A}|$ で割り左から $\hat{A}^{-1}$ を掛けると

$$\hat{A} = \frac{\hat{\tilde{A}}}{|\hat{A}|} \tag{220}$$